EFFECTIVE RHEOLOGICAL PROPERTIES OF A DISPERSE MIXTURE OF VISCOELASTIC MATERIALS

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The article deals with the problem of determining the compressive and shear moduli, and also the shear and bulk viscosity of a viscoelastic medium with spherical inclusions of some other viscoelastic material. Calculations are presented for an elastic medium with inclusions of a viscous liquid.

When the linear scale of the fields of mean strains and stresses in a composite material greatly exceeds its inner structural scale characterizing the dimensions and disposition of inhomogeneities, then it is natural to describe the process of deformation within the framework of a continual approximation. Then two fundamental problems arise: derivation of the averaged equations and their closure, i.e., obtaining notions for the effective characteristics of the material under examination.

The mentioned continual equations are usually postulated phenomenologically or they are obtained by spatial averaging, and their closure is effected on the basis of independent methods reviewed in [1-3]. Most of these methods suggested for elastic materials with discrete inclusions are based on variants of the semiempirical cell model [4] on the self-consistent model [5]. A strict analysis with low bulk concentration of the discrete component was carried out in [6] by the method of summation of binary interactions between the inclusions, which was suggested for the first time in [7] for the problem of settlement of suspensions. Out of the recent attempts to construct a relatively complete theory for arbitrary concentrations of inclusions we mention [8]; a general discussion of the problem in connection with the continuum mechanics of inhomogeneous elastic media is contained in [9]. Generalization of these results to materials with viscoelastic components is usually carried out on the basis of the so-called "elastic—viscoelastic analogy" [3]; some conclusions that are useful in this connection are presented in [10-13].

The solution of both mentioned problems by one schema is possible with the use of the method of averaging over the ensemble of physically admissible configurations of the system of inclusions in combination with methods of the theory of self-consistent fields. Such methods were worked out and successfully used for solving analogous problems in the hydrodynamics of incompressible suspensions [14] and in the theory of heat and mass transfer in disperse media [15]; as far as the present authors know, these methods had previously not been used for the analysis of solid deformed materials.

Averaging over a configuration ensemble has the following fundamental advantages. Firstly, the continual equations are derived and closed by the same methods, i.e., it is not necessary to invent special models for determining the effective properties. Secondly, there is no need of additional hypotheses on the correlation of the results of averaging over spatial objects of a different nature — volume and variously oriented facets. Thirdly, there is no need to demand that a large number of inclusions be contained in a small physical volume, and the solution of the continual equations describes the most probable state of the material.

Below we investigate the dynamic behavior of a composite material consisting of a continuous viscoelastic matrix and spherical inclusions of some other viscoelastic material distributed in the matrix. The properties of material of a special type were studied in more detail: of an elastic medium with inclusions containing a Newtonian viscous liquid; such material may be regarded as the model of real filled porous strata.

General Theory. We will describe the mechanical behavior of the matrix and of the material in the inclusions with the aid of a single equation

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$$D\partial^2\mathbf{U}/\partial t^2 = \nabla\boldsymbol{\Sigma}, \tag{1}$$

with small deformations the stress tensor is expressed through the strain tensor and the displacement vector as follows:

$$\boldsymbol{\Sigma} = B\mathbf{I}\,\mathrm{div}\,\mathbf{U} + 2\Gamma\left(\mathbf{E} - \frac{1}{3}\mathbf{I}\,\mathrm{div}\,\mathbf{U}\right), \tag{2}$$

and the components $\mathbf{E}$ are represented in the ordinary way through the derivatives of the components $\mathbf{U}$ with respect to the coordinates.

The true values of B and Γ in (2) for viscoelastic materials may be regarded as operators.

$$B = K + Z\frac{\partial}{\partial t}, \quad \Gamma = M + H\frac{\partial}{\partial t}. \tag{3}$$

The displacement vector $\mathbf{U}$, the stress tensor $\boldsymbol{\Sigma}$ and the strain tensor $\mathbf{E}$, density D, and also the coefficients in (3) are generalized functions, continuous in the matrix and in the inclusions but possibly having discontinuities on interfaces. They can be easily written if we introduce the characteristic function θ, which is equal to unity at points occupied by the matrix, and equal to zero inside the inclusions. For instance:

$$K = k_0\theta + k_1(1-\theta), \quad Z = \xi_0\theta + \xi_1(1-\theta),$$
$$M = \mu_0\theta + \mu_1(1-\theta), \quad H = \eta_0\theta + \eta_1(1-\theta).$$

For purely elastic material $\xi = \eta = 0$, and k and μ are the bulk modulus and the shear modulus, respectively; for a compressible Newtonian liquid $\mu = 0$, k is the modulus of compressibility, ξ and η are the coefficients of bulk and shear viscosity, respectively.

The problem consists in obtaining the following equations from (1) and (2) with a view to formulas (3): equations for the mean displacement vector $\mathbf{u}$ and the mean stress tensor $\boldsymbol{\sigma}$ and the mean strain tensor $\boldsymbol{\varepsilon}$, formally associated with some homogeneous continuum modeling the composite material, and also in calculating all the coefficients appearing in such equations as functions of the moduli of elasticity and coefficients of viscosity of the materials of the phases, of the concentration of the inclusions and of structural parameters characterizing the packing of the inclusions. We point out that we do not deal here with the relative motion of phases or components of the material; this limitation is important for suspensions [16].

The mentioned equations can be obtained by averaging (1) and (2) with respect to the distribution function of the positions of the centers of the spherical inclusions which fully describes their configurational ensemble; the corresponding mathematical apparatus was developed in [14]. Omitting intermediate considerations, we present here these equations, and also the formal relations for the magnitudes they contain, which characterize the effective properties of the composite material, and we also present the formulation of the problem of an isolated (test) inclusion whose solution enables us to express the mentioned magnitudes explicitly. It is expedient to apply first the Fourier transform with respect to time to (1)-(3); this leads to the replacement of the operator ∂/∂t by the factor iω; in the subsequent analysis the same designations were used for the transformations of the variables as for the corresponding originals.

Averaging the transformed equations (1) and (2) over the configurational ensemble leads to the equations

$$-d\omega^2\mathbf{u} = \nabla\boldsymbol{\sigma}, \quad d = d_1\rho + d_0(1-\rho),$$
$$\boldsymbol{\sigma} = \beta\mathbf{I}\,\mathrm{div}\,\mathbf{u} + 2\gamma\left(\boldsymbol{\varepsilon} - \frac{1}{3}\mathbf{I}\,\mathrm{div}\,\mathbf{u}\right), \tag{4}$$

where ε is expressed in the ordinary way through the derivatives of u, and the following magnitudes are introduced:

$$\beta = k + i\omega\xi, \quad \gamma = \mu + i\omega\eta. \tag{5}$$

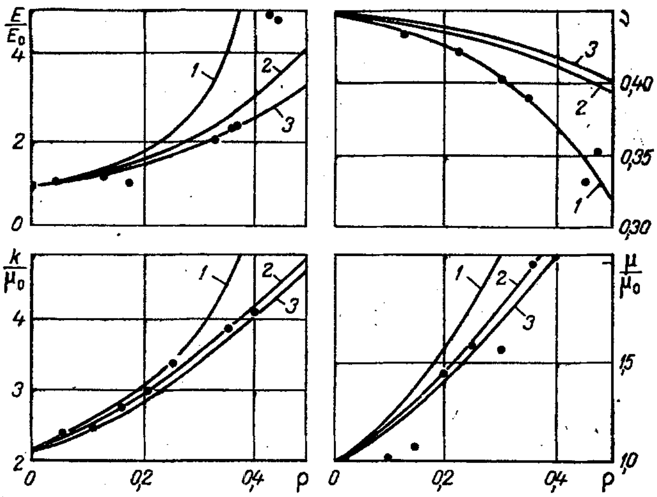

Fig. 1. Dependence of the relative modulus of elasticity E/E_0 and of the Poisson ratio ν for the experimental conditions in [21] ($E_1/E_0 = 40.8$, $\nu_0 = 0.45$, $\nu_1 = 0.21$) and of the relative compressive and shear moduli for the experimental conditions in [22] ($k_0/\mu_0 = 2.2$, $k_1/\mu_0 = 32$, $\mu_1/\mu_0 = 24.3$) on the bulk concentration of the spherical inclusions: 1) Eqs. (9) and Hill's theory [5]; 2) Eqs. (11); 3) theory of Kerner–Levin [8, 19]; dots) experimental data.

Here, k, μ are the bulk and shear modulus, respectively; ξ, η are the coefficients of bulk and shear viscosity, respectively, referred to the heterogeneous material as a whole. In principle they can be determined from the relations [14-16]

$$(\beta-\beta_0)\operatorname{div}\mathbf{u} = (\beta_1-\beta_0)\,n\int\limits_{r\leqslant a}\operatorname{div}\mathbf{u}^*d\mathbf{r},$$
$$(\gamma-\gamma_0)\left(\boldsymbol{\varepsilon}-\frac{1}{3}\mathbf{I}\operatorname{div}\mathbf{u}\right) = (\gamma_1-\gamma_0)\,n\int\limits_{r\leqslant a}\left(\boldsymbol{\varepsilon}^*-\frac{1}{3}\mathbf{I}\operatorname{div}\mathbf{u}^*\right)d\mathbf{r}, \tag{6}$$

which essentially are equivalent to the formulas obtained in [17]. Integration here is carried out with respect to the volume of a single test inclusions with the center at the point r = 0, and the asterisk as superscript indicates the magnitudes obtained by averaging with respect to the nominal distribution function of the positions of the centers of other inclusions corresponding to the configurations in which the position of the test inclusion is fixed.

For nominal means with one fixed inclusion we can easily obtain equations of the same form as (4). However, the coefficients in them, which replace β and γ from (5), have to be determined from relations type (6) in which the integrands contain magnitudes obtained by averaging with respect to the ensemble of configurations with fixed disposition of the centers of two inclusions [14]. By continuing this process we can obtain an infinite chain of mutually connected equations for the nominal means corresponding to the ensemble of configurations in which the position of the centers of different numbers of inclusions are fixed. As a result there arises the problem of terminating or closing this chain; it is analogous in meaning to the same problem in the physics of dense gases and liquids or in the statistical theory of turbulence. Constructive closure can be effected on the basis of the hypothesis of the proportionality of the effect of inclusions on the physical properties of the medium near the test particle to their nominal mean bulk concentration ρ^* which asymptotically tends to ρ with increasing distance to the test inclusion [14, 15]. Then outside the test particle we may write

$$\mathbf{u}^{(1)} = \mathbf{u}+\mathbf{u}',\ \boldsymbol{\varepsilon}^{(1)} = \boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}',\ \boldsymbol{\sigma}^{(1)} = \boldsymbol{\sigma}+\boldsymbol{\sigma}',$$
$$\boldsymbol{\sigma}' = \beta'\mathbf{I}\operatorname{div}\mathbf{u}'+2\gamma'\left(\boldsymbol{\varepsilon}'-\frac{1}{3}\mathbf{I}\operatorname{div}\mathbf{u}'\right),$$

where the tensor ε' is correlated with the derivatives of u' by an ordinary linear relation, σ is expressed through ε and u in accordance with (4), and β' and γ' can be represented in the form

$$\beta' = \beta_0+(\beta-\beta_0)\frac{\rho^*}{\rho},\ \gamma' = \gamma_0+(\gamma-\gamma_0)\frac{\rho^*}{\rho},\ \rho^* = n\int\limits_{r\leqslant a}\varphi(\mathbf{r})\,d\mathbf{r}.$$

The function $\varphi(\mathbf{r})$ is expressed in the standard way through the binary distribution function of the inclusions characterizing the structure of the examined heterogeneous material [14].

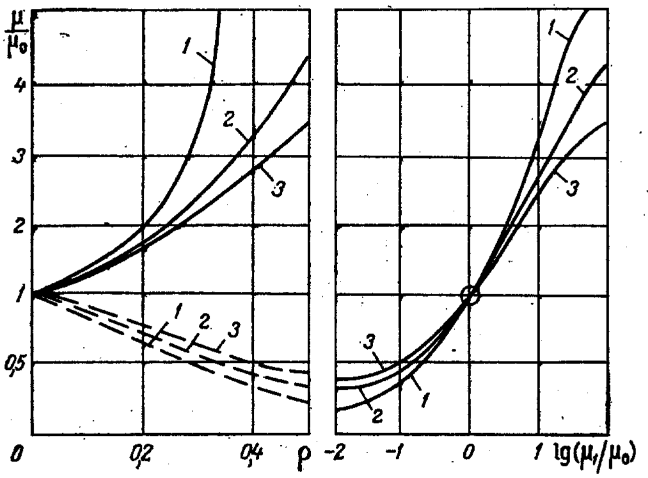

Fig. 2. Dependences of the relative shear moduli μ/μ_0 for weakly compressible elastic composites on ρ (solid and dashed lines correspond to $\mu_1/\mu_0 = 10^{-2}$ and 10^2, respectively) and on μ_1/μ_0 for $\rho = 0.5$; the notation of 1-3 is the same as in Fig. 1.

The boundary-value problem of the test inclusion has the form

$$\begin{gathered} -d'\omega^2\mathbf{u}' = \nabla\sigma',\ \mathbf{r} > a;\ -d_1\omega^2\mathbf{u}^{(2)} = \nabla\sigma^{(2)},\ r < a; \\ d' = d_1\rho^* + d_0(1-\rho^*);\ \mathbf{u}',\ \boldsymbol{\varepsilon}' \to 0,\ r \to \infty; \\ \mathbf{u} + \mathbf{u}' = \mathbf{u}^{(2)},\ n(\sigma + \sigma') = \mathbf{n}\sigma^{(2)},\ r = a;\ \mathbf{u}^{(2)},\ \boldsymbol{\varepsilon}^{(2)} < \infty,\ r = 0. \end{gathered} \tag{7}$$

Here, n is the unit vector of the normal to the surface of the inclusion, and the stress tensors are correlated with the corresponding strain tensors and displacement vectors by the relations presented above. The primes denote magnitudes describing distortions introduced by the test inclusion into the averaged fields u, ε, and σ. The tensor components σ and ε are determined at a point occupied by the center of the test inclusion, as if this inclusion were not there at all, the in solving problem (7) they may be regarded as specified. The magnitude $d' = d_1\rho^* + d_0(1 - \rho^*)$, and $u^{(2)}$ and $\varepsilon^{(2)}$ by definition coincide with u^* and ε^* contained in (6). Formally (7) is a boundary-value problem of the deformation of a spherical particle in a fictitious medium whose properties depend in a certain way (dictated by the form of the function $\varphi(r)$ characterizing the packing of the inclusions) on the position of the point relative to the surface of the particle, with specified state of strain while moving away from it. The solution of this problem depends on the values of k, μ, ξ, and η, as well as on the parameters. Using this solution in the calculation of the integrals in (6), we arrive at a system of transcendental equations with respect to these magnitudes whose solution makes it possible to find them as functions of known moduli and coefficients of viscosity of the materials of the matrix and of the inclusions, and also of ρ and of the parameters used in the determination of the function $\varphi(r)$. It is clear from the form of Eq. (4) that the mentioned magnitudes fully describe the elastic and ductile properties of the examined type of composite material in continual approximation.

The function $\varphi(r)$ depends on details of the packing of the inclusions, and its determination under various conditions represents a complex problem of statistical physics; within the framework of the theory developed here, this function has to be regarded as known. For isotropic materials with random distribution of the inclusions we can use directly the results of the classical theory of dense gases and liquids, determining $\varphi(r)$ on the basis of the known approximations of Kirkwood, the hyperchain approximation, the Perkus-Jevik approximation, as was done, e.g., in [18] in the calculation of effective ductility and thermal conductivity. The simplest results are obtained when the effect of the test inclusion on the disposition of the other inclusions is altogether neglected. This is permissible for materials with relatively small bulk concentration of the inclusions, and it corresponds formally to the model in which $\varphi(r) = 1$ and $\rho^* = \rho$, i.e., the test inclusion is submerged in a homogeneous fictitious medium whose properties coincide with the corresponding properties of the material as a whole. In this case we thus arrive at a simple variant of the self-consistent theory which was suggested for the first time in [5] on the basis of empirical considerations. The use of an analogous model in the hydrodynamics of suspensions and in the theory of heat and mass transfer led to satisfactory results up to and including values of $\rho \approx 0.20$–0.25 [16, 18].

As the subsequent approximation, also suitable for concentrated systems, it is natural to consider the discontinuous function $\rho^* = 0$ for $a < r < 2a$ and $\rho^* = \rho$ for $r > 2a$; this leads to a model in which the fictitious medium with homogeneous properties is separated from the

TABLE 1. Effective Relative Viscosity η/η_0 of Monodisperse Suspension of Rigid Spheres in an Incompressible Viscous Liquid According to Different Models

No. of model	ρ				
	0,1	0,2	0,3	0,4	0,5
1	1,3	2,7	4,0	∞	—
2	1,3	1,6	2,1	2,6	3,5
3	1,3	1,7	2,3	3,2	4,6
4	1,3	1,7	2,4	3,5	6,0
5	1,3	1,8	2,7	4,5	—

Note: 1) $\eta/\eta_0 = (1 - 5\rho/2)^{-1}$, which corresponds to the first formula in (10) and to Hill's theory [5]; 2) theory from [8, 19]; 3) model with free concentric layer (Eq. (11)); 4) numerical calculation in [18] for a model with uniform distribution of the centers of particles outside a sphere with radius $2a$, concentric with the test particle; 5) numerical calculation in [18] for Kirkwood's superposition model.

surface of the test inclusion by a concentric layer with thickness a filled with pure material of the matrix. It seems that this type of model was suggested for the first time in [19], then it was dealt with by many authors [2, 3] but the thickness of the layer remained undetermined.

Further refinement entails the use of stricter notions for $\varphi(r)$. In this case the properties of the fictitious medium surrounding the test inclusion prove to be dependent on the distance to its surface, and in solving problem (7) it is indispensable to use numerical methods [18]. We point out that the suggested general schema is not only suitable for investigating randomized, but also ordered structures if the function (r) is determined in the corresponding manner.

Effective Properties. Continual description of the deformation processes of heterogeneous media is justified if the characteristic scale λ of the mean stress and strain fields is much larger than the scale of the internal structure which for the type of media under examination with the use of ensemble averaging may be identified with the size a of the inclusions. Thus, we have the indispensable condition $\lambda \gg a$. In dynamic deformation of elastic media λ is equal in order of magnitude to the length of the longitudinal or transverse elastic waves c/ω, where $c \sim (\{k, \mu\}/d)^{1/2}$. For nonsteady processes in viscous liquids, λ has the order of the characteristic length of viscous attenuation of the pulse $(\{\xi, \eta\}/d\omega)^{1/2}$. Hence the conditions of adequate continual approximation are clear. Below we require that these conditions also be fulfilled for the materials of the separate phases, i.e., we take it that the following strong inequalities are correct:

$$\omega^2 \ll \frac{\{k_j, \mu_j\}}{d_j a^2}, \quad |\omega| \ll \frac{\{\xi_j, \eta_j\}}{d_j a^2}, \quad j = 0, 1. \tag{8}$$

If for any of the magnitudes k_j, μ_j, ξ_j or η_j ($j = 0, 1$) the inequality in (8) does not apply, then the corresponding term on the right-hand sides of the equations in (7) has to be discarded to avoid excessive accuracy. Physically these inequalities correspond to the negligible effect of the forces of inertia on the deformation process. The first group of inequalities in (8) applies to solids and to compressible dropping liquids practically at all frequencies that are of interest. An idea of the feasibility of the second group if inequalities can be obtained by examining, as an example, a water-saturated ($\nu \approx 0.01$ cm²/sec) porous body with the characteristic pore dimension 0.001 cm, where it follows from (8) that $|\omega| \ll 10^4$ rad. We note that inequalities analogous to (8) need not be fulfilled when a is replaced by the macroscopic scale λ.

Within the framework of the simplest model, where the test particle is submerged in a homogeneous fictitious medium, problem (7) reduces to the well-known problem of distortions introduced by a spherical particle into the state of uniform strain of a continuum. Using the solution of this problem, we obtain from (6) after some calculations the equations for the complex moduli β and γ:

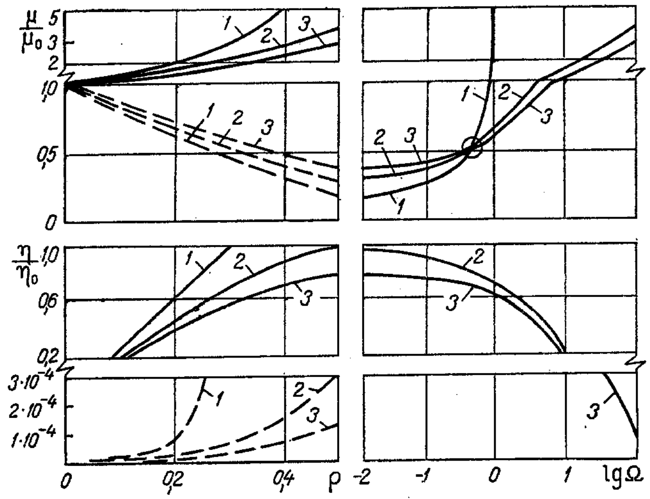

Fig. 3. Relative shear moduli and shear viscosities of an elastic matrix with cavities filled with a viscous liquid as functions of ρ (solid and dashed curves correspond to $\Omega = 10^{-2}$ and 10^{2}, respectively) and on Ω for $\rho = 0.5$; the notation of 1-3 is the same as in Fig. 1

$$\gamma = \gamma_0 + (\gamma_1 - \gamma_0)\frac{15(\beta + 4\gamma/3)}{9\beta + 8\gamma + 6(\beta + 2\gamma)(\gamma_1/\gamma)}\rho,$$
$$\beta = \beta_0 + (\beta_1 - \beta_0)\frac{\beta + 4\gamma/3}{\beta_1 + 4\gamma/3}\rho, \tag{9}$$

which formally coincide with those obtained in [5], where elastic materials only were dealt with. These equations are substantially simplified for weakly compressible materials ($|\beta| \gg |\gamma|$), when

$$\gamma = \frac{1}{6}\{\gamma_0(3-5\rho) - \gamma_1(2-5\rho) + [(\gamma_0(3-5\rho) - \gamma_1(2-5\rho))^2 + 24\gamma_0\gamma_1]^{1/2}\},\ \beta = \frac{\beta_0\beta_1}{\beta_0\rho + \beta_1(1-\rho)}. \tag{10}$$

This last relation coincides formally with an analogous formula in [19]. For $\rho \ll 1$, we can easily obtain the following from (9) with an accuracy to the terms of the first order with respect to ρ inclusively:

$$\gamma = \gamma_0 + (\gamma_1 - \gamma_0)\frac{15(\beta_0 + 4\gamma_0/3)}{9\beta_0 + 8\gamma_0 + 6(\beta_0 + 2\gamma_0)(\gamma_1/\gamma_0)}\rho,$$
$$\beta = \beta_0 + (\beta_1 - \beta_0)\frac{\beta_0 + 4\gamma_0/3}{\beta_1 + 4\gamma_0/3}\rho.$$

It can be shown that in special cases the well-known Einstein formulas follow from this for the viscosity of incompressible dilute suspensions, the formulas of Oldroyd, Mackenzie, and Hashin for the elastic moduli of composite material with elastic matrix and viscous inclusions, and some others [20].

Relations (9) and (10) are approximately correct only for moderately concentrated systems. For mixtures with higher concentration it is natural to use as reasonable approximation a model according to which the test particle is separated from the homogeneous fictitious medium by a concentric layer of thickness a filled with pure material of the matrix. In that case we obtain from (7) a three-layer boundary-value problem whose solution is also given in [3]. In particular, inside the test inclusion the components of the displacement vector without the angle parts are

$$u_r^{(2)} = u_r = A_1 r - \frac{6\nu_1}{1-2\nu_1}A_2 r^3,\ u_\theta^{(2)} = u_\theta^* = A_1 r - \frac{7-4\nu_1}{1-2\nu_1}A_2 r^3,$$

where the arbitrary constants A_j are determined together with the constants contained in the expressions for the components of the displacement vector inside and outside the layer $a < r < 2a$ from the system of algebraic equations that follow from the boundary conditions for $r = a$ and $r = 2a$ (this system is too cumbersome to be presented here). In our case we obtain from (6) the equations

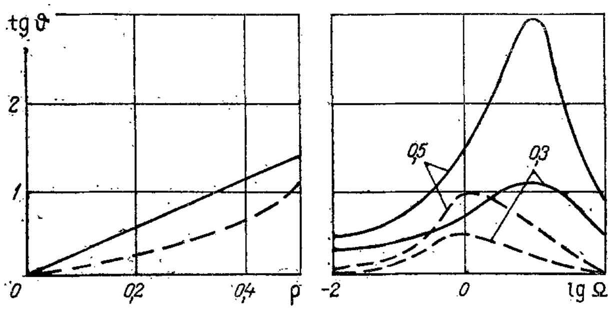

Fig. 4. Dependences of the tangent of the loss angle tan ϑ of the transverse and longitudinal elastic waves (dashed and solid curves, respectively) on ρ for Ω = 1 and on Ω for ρ = 0.3 and 0.5 (the numbers next to the curves).

$$\gamma = \gamma_0 + (\gamma_1 - \gamma_0)\left[A_1 - \frac{21}{5}\left(\beta_1 + \frac{1}{3}\gamma_1\right) a^2 A_2\right]\rho,$$

$$\beta = \beta_0 + (\beta_1 - \beta_0)\left[\left(\beta_0 + \frac{4}{3}\gamma_0\right)\left(\beta + \frac{4}{3}\gamma\right) - \frac{3}{2}(\beta - \beta_0)(\gamma - \gamma_0)\right]\rho \times \tag{11}$$

$$\times\left[\left(\beta_0 + \frac{4}{3}\gamma_0\right)\left(\beta_1 + \frac{4}{3}\gamma\right) + \frac{3}{2}(\beta_1 - \beta_0)(\gamma - \gamma_0)\right]^{-1}.$$

Although for finding β and γ in accordaince with (11), only the constants A_j are needed, determining the latter requires the mentioned system of algebraic equations to be completely solved; this is most simply done numerically. We note that relations (9) and (10) are equally correct for monodisperse as well as polydisperse mixtures with moderate concentration. On the other hand, Eqs. (11) are based on a model which substantially uses the assumption that the spherical inclusions are equal in size. Therefore, although the results for β and γ applied to a monodisperse mixture do not depend on the size of the inclusions, their applicability to polydisperse mixtures with large scatter of the radii of the inclusions appears, generally speaking, doubtful.

Figures 1 and 2 present the results of analytical and numerical calculations of the effective properties of composite material consisting of an elastic matrix and elastic inclusions, and also experimental data from [21, 22]. It can be seen that the results of the suggested theory occupy an intermediate position in relation to those following from Hill's theory [5] described by formulas (9) and (10) and those following from Kerner's theory [19] which for materials with spherical inclusions coincide with Levin's theory [8]. Hill's theory, which is correct for moderately concentrated materials only, describes the experimental values of the Poisson ratio from [21] better even at high concentrations. This artifact is apparently due to the fact that the system of inclusions in the materials dealt with in [21] was substantially polydisperse.

It is difficult to arrive at a definite conclusion on the basis of Fig. 1 as to which of the theories corresponds best to the experiments. To find an answer to this question, we carried out analogous calculations of the effective relative viscosity of monodisperse suspensions of rigid spheres in an incompressible Newtonian liquid, for which there are many experimental data available. The values of relative viscosity obtained in [18] with the aid of the numerical solution of problem (7) agree very well with the experimental values if ρ^* in the vicinity of the test particle is determined on the basis of Kirkwood's superposition model; the agreement is somewhat poorer if we assume uniform distribution of the centers of the inclusions outside the forbidden sphere $r = 2a$ (see Table 1). It can be seen that the results obtained in [18] and here on the basis of an approximate model with a homogeneous concentric layer are considerably better than those following from the theory of [8, 19], and this supports the choice in favor of the theory suggested here.

The developed theory makes it possible to evaluate the characteristics of a disperse composite material in a broad range of change of the viscoelastic properties of its components. For the sake of determinacy we will examine in somewhat greater detail the effective properties of a material consisting of an elastic matrix ($\xi_0 = \eta_0 = 0$) and of spherical cavities randomly distributed in it and filled with a viscous liquid ($\mu_0 = 0$) which may be regarded as the simplest model of a porous stratum filled with a liquid.

If we separate in Eqs. (9) the real and imaginary parts, we obtain a system of four transcendental equations for determining the values of k, μ, ξ, and η describing moderately concentrated material. On the assumption that the elastic matrix and the viscous liquid in the inclusions are only slightly compressible, we obtain after calculations for the shear modulus and the shear viscosity of such material from (10):

$$\frac{\mu}{\mu_0}=\frac{1}{6}(R\cos\psi+R'\cos\psi'),\quad \frac{\eta}{\eta_1}=\frac{1}{6\Omega}(-R\sin\psi+R'\sin\psi'), \tag{12}$$

where the following dimensionless magnitudes were introduced:

$$\Omega=\omega\eta_1/\mu_0,\quad R=[(3-5\rho)^2+\Omega^2(2-5\rho)^2]^{1/2},$$

$$R'=\{[(3-5\rho)^2-\Omega^2(2-5\rho)^2]^2+[24-2(3-5\rho)(2-5\rho)]^2\Omega^2\}^{1/4},$$

$$\psi=\frac{1}{2}\operatorname{arctg}\frac{2-5\rho}{3-5\rho}\Omega,\quad \psi'=\frac{1}{2}\operatorname{arctg}\frac{24-2(3-5\rho)(2-5\rho)}{(3-5\rho)^2-\Omega^2(2-5\rho)^2}\Omega.$$

In limit cases of low-frequency and high-frequency processes of deformation we obtain hence

$$\frac{\mu}{\mu_0}\to 1-\frac{5}{3}\rho,\quad \frac{\eta}{\eta_1}\to\frac{25}{3}\,\frac{\rho(1-\rho)}{3-5\rho},\quad \Omega\to 0;$$

$$\frac{\mu}{\mu_0}\to\left(1-\frac{5}{2}\rho\right)^{-1},\quad \frac{\eta}{\eta_1}\to\frac{25}{4}\,\frac{\rho}{\Omega^2}\left(1-\frac{5}{2}\rho\right)^{-2},\quad \Omega\to\infty$$

(it is easy to show that the values $\Omega \gg 1$ are possible without loss of correctness of inequality (8)).

The results of the calculations on the basis of formulas (12), and also according to the theory suggested here (formulas (11)) and by the model in [8, 19] with the use of the elastic–viscoelastic analogy [3] are presented in Fig. 3. This type of data proves to be very important in the analysis of parameters of longitudinal and transverse linear elastic waves propagating in porous media filled with a liquid. The mentioned parameters are absolutely indispensable in evaluating the characteristics of such media on the basis of the results of some geophysical investigations. Cavities filled with viscous liquid lead to substantial dispersion of the speeds of elastic waves and to their effective attenuation. We cannot dwell in detail on these problems but we present in Fig. 4 the concentration and frequency dependences of the tangent of the loss angle of the longitudinal and transverse waves in a porous medium with spherical cavities filled with a liquid. As was to be expected, absorption increases monotonically with increasing concentration of inclusions, and it is characterized by distinct maxima at certain frequencies.

The physical cause of such absorption of elastic-wave energy lies in the viscous dissipation of the energy of circulatory flows inside the cavities which are excited by the wave process. In real porous media there may also act another mechanism which will not be dealt with here: it is connected with the relative phase shift upon wave superposition. Evaluations show that under different conditions each of the mentioned mechanisms of dissipation may predominante.

In conclusion we will deal with the problem of introducing the usually used relaxation times and time of elastic aftereffect for characterizing a viscoelastic mixture. It can be easily seen that this can be reasonably done only where the description of low-frequency processes is concerned, where we may confine ourselves with high accuracy to a few of the first terms in the expansions of β and γ in degrees of $i\omega$. With an accuracy up to terms of second order included, ξ and η have to be calculated for $\omega = 0$, and k and μ have to be expressed as $k^{(0)} - \omega^2 k^{(2)}$ and $\mu^{(0)} - \omega^2\mu^{(2)}$ with coefficients not dependent on ω. Then, applying an inverse Fourier transformation to the last relation in (4), we obtain the rheological equation of state of a viscoelastic mixture in the usual form

$$\boldsymbol{\sigma}=k^{(0)}\left(1+T_k^{(1)}\frac{\partial}{\partial t}+T_k^{(2)2}\frac{\partial^2}{\partial t^2}\right)\mathbf{I}\operatorname{div}\mathbf{u}+2\mu^{(0)}\left(1+T_\mu^{(1)}\frac{\partial}{\partial t}+T_\mu^{(2)2}\frac{\partial^2}{\partial t^2}\right)\left(\boldsymbol{\varepsilon}-\frac{1}{3}\mathbf{I}\operatorname{div}\mathbf{u}\right),$$

$$T_k^{(1)}=\frac{\xi}{k^{(0)}},\quad T_k^{(2)2}=\frac{k^{(2)}}{k^{(0)}},\quad T_\mu^{(1)}=\frac{\eta}{\mu^{(0)}},\quad T_\mu^{(2)2}=\frac{\mu^{(2)}}{\mu^{(0)}},$$

where $T_k^{(j)}$ and $T_\mu(j)$ are the characteristic times of elastic aftereffect. If we confine ourselves to an accuracy up to the first order with respect to $i\omega$, we have to discard in the presented equation the terms with the second derivatives with respect to time.

For dilute systems the equations of state with a finite number of relaxation times and times of aftereffect may be obtained without imposing constraints on the frequency of the processes under examination. In fact, in this case the complex moduli β and γ are expressed as special ones of dividing the polynomials with $i\omega$ not higher than second order. The last relation in (4) may be increased to polynomials wthat are in the denominators of β and γ, after which is is easy to apply the inverse Fourier transformation. For instance, for incompressible materials $\sigma = 2\gamma\varepsilon$, whereby

$$\gamma = \gamma_0 \left[1 + \frac{5(\gamma_1 - \gamma_0)}{3\gamma_0 + 2\gamma_1} \rho\right],$$

so that

$$(3\gamma_0 + 2\gamma_1)\sigma = 2\gamma_0 [(3 - 5\rho)\gamma_0 + (2 + 5\rho)\gamma_1] \varepsilon.$$

Applying the inverse Fourier transformation, we obtain for suspensions of elastic spheres in a Newtonian liquid:

$$\left(1 + \frac{3}{2}\frac{\eta_0}{\mu_1}\frac{\partial}{\partial t}\right)\sigma = 2\eta_0\left(1 + \frac{5}{2}\rho\right) \times \left(1 + \frac{3 - 5\rho}{2 + 5\rho}\frac{\eta_0}{\mu_1}\frac{\partial}{\partial t}\right)\frac{\partial \varepsilon}{\partial t},$$

and for an elastic matrix with cavities filled with a liquid:

$$\left(1 + \frac{2}{3}\frac{\eta_1}{\mu_0}\frac{\partial}{\partial t}\right)\sigma = 2\mu_0\left(1 - \frac{5}{3}\rho\right) \times \left(1 + \frac{2 + 5\rho}{3 - 5\rho}\frac{\eta_1}{\mu_0}\frac{\partial}{\partial t}\right)\varepsilon.$$

In regard to their form these relations are like the equations of state by Frölich–Zak and Oldroyd [20]; however, the relaxation time and the time of elastic aftereffect in them are different.

We note that in principle it is easy to generalize the developed theory to mixtures whose components obey linear equations of state differing in form from the simplest equations in (2).

NOTATION

a, Radius of particles (inclusions); D, detail density; d, ordinary density; E, Young's modulus; I, unit tensor; K, detail bulk modulus; k, ordinary bulk modulus; n, number concentration of particles; n, unit vector of the normal to the surface of the test particle; R, R', coefficients in (12); r, radial coordinate; $T_k^{(j)}$, $T_\mu(j)$, times of elastic aftereffect; t, time; U, detail displacement vector; u, mean displacement vector; Β, Γ, operators in (2); β, complex modulus of compression of the mixture; γ, complex shear modulus of the mixture; λ, macroscopic linear scale; M, detail shear modulus; μ, ordinary shear modulus; ν, Poisson ratio and kinematic viscosity; Z, ξ and H, η, detail and ordinary coefficients of bulk and shear viscosity, respectively; Ε, ε and Σ, σ, detail and mean strain and stress tensors, respectively; θ, characteristic function; ρ, bulk concentration of particles; ρ*, bulk concentration in the vicinity of the test inclusion; ψ, ψ', angles in (12); ω, Ω, dimensional and dimensionless angular frequency, respectively; subscripts 0 relate to the matrix, subscripts 1 to inclusions, an asterisk to fields inside the test inclusion; $\varphi(r)$, binary distribution function of spherical inclusions.

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